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# On a Generalization of Continued Fractions and the Estimation of Some Integrals*

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## 1. INTRODUCTION

In various parts of mathematical physics one encounters integrals of the form

$$I(a) = \int_0^\infty t^n e^{-t-at^k}\, dt, \tag{1}$$

where $a > 0$ and $n$ and $k$ are positive integers. The perturbation expansion

$$\begin{aligned} f(a) &= \int_0^\infty t^n e^{-t} \left[ \sum_{m=0}^\infty (-at^k)^m/m! \right] dt \\ &= \sum_{m=0}^\infty \frac{(-a)^m}{m!} \int_0^\infty t^{n+mk} e^{-t}\, dt \end{aligned} \tag{2}$$

is an asymptotic series divergent for all $a > 0$. Although useful, it is not easy to obtain estimates of the error involved in using its partial sums.

In this note we wish to describe a method based upon an extension of the classical theory of continued fractions which yields upper and lower bounds. For the case $k = 2$, the method is the conventional method.

## 2. THE CASE $k = 2$

Let us consider the case $k = 2$ first where $I(a)$ is reducible to the exponential integral. Write

$$u_n = \int_0^\infty t^n e^{-t} e^{-at^2}\, dt, \qquad n = 0, 1, 2, \ldots . \tag{2.1}$$

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Then, for $n \geqslant 2$,

$$
\begin{aligned}
u_n &= \int_0^\infty (t^{n-1}e^{-t})(te^{-at^2})\,dt \\
&= (t^{n-1}e^{-t})(-e^{-at^2}/2a)]_0^\infty + \frac{1}{2a}\int_0^\infty [-t^{n-1}e^{-t} + (n-1)\,t^{n-2}e^{-t}]\,e^{-at^2}\,dt.
\end{aligned} \tag{2.2}
$$

Hence,

$$
\begin{aligned}
u_n &= \frac{1}{2a}[(n-1)\,u_{n-2} - u_{n-1}], \\
u_{n-2} &= \frac{u_{n-1}}{n-1} + \frac{2au_n}{n-1}.
\end{aligned} \tag{2.3}
$$

Thus,

$$
\frac{u_{n-2}}{u_{n-1}} = \frac{1}{n-1} + \frac{2a}{(n-1)\,u_{n-1}/u_n}. \tag{2.4}
$$

Hence, if we set $z_n = u_n/u_{n+1}$, we have

$$
z_n = \frac{1}{n+1} + \frac{2a}{(n+1)\,z_{n+1}}, \qquad n = 0, 1, 2,\ldots. \tag{2.5}
$$

For $n = 1$, we have

$$
\begin{aligned}
u_1 &= \int_0^\infty te^{-t}e^{-at^2}\,dt \\
&= (e^{-t})\left(\frac{-e^{-at^2}}{2a}\right)\Big]_0^\infty - \frac{1}{2a}\int_0^\infty e^{-t}e^{-at^2}\,dt \\
&= \frac{1}{2a} - \frac{u_0}{2a}
\end{aligned} \tag{2.6}
$$

Hence, if we can obtain bounds for $u_0/u_1$ from (2.5), we can use these bounds in (2.6) to obtain bounds for $u_0$, and thus for any $u_n$.

## 3. Upper and Lower Bounds

We can employ (2.5) to obtain bounds in the usual fashion. We have clearly

$$
z_n > \frac{1}{n+1}, \qquad n = 0, 1, 2,\ldots, \tag{3.1}
$$

whence, using (5) again

$$z_n < \frac{1}{n+1} + \frac{2a}{(n+1)/(n+2)}, \qquad n = 0, 1, \ldots . \tag{3.2}$$

It follows that

$$z_n > \frac{1}{n+1} + \frac{2a}{(n+1)\left[\dfrac{1}{(n+2)} + \dfrac{2a}{(n+2)/(n+3)}\right]}, \tag{3.3}$$

and so on. Using these results for $n = 0$, we obtain the required estimates for $u_0/u_1$.

## 4. The Case $k = 3$

Consider next the case $k = 3$. The general case, $k = 4, 5, \ldots$, can be handled in similar fashion. Write

$$u_n = \int_0^\infty t^n e^{-t} e^{-at^3}\, dt. \tag{4.1}$$

Then for $n \geqslant 3$ we have

$$\begin{aligned} u_n &= \int_0^\infty t^{n-2} e^{-t} t^2 e^{-at^3}\, dt \\ &= \int_0^\infty \frac{e^{-at^3}}{3a} [(n-2)\, t^{n-3} e^{-t} - t^{n-2} e^{-t}]\, dt \\ &= \frac{1}{3a} [(n-2)\, u_{n-3} - u_{n-2}]. \end{aligned} \tag{4.2}$$

Hence,

$$u_{n-3} = \frac{u_{n-2}}{(n-2)} + \frac{3au_n}{(n-2)}, \tag{4.3}$$

and

$$\frac{u_{n-3}}{u_{n-2}} = \frac{1}{(n-2)} + \frac{3a}{(n-2)\, u_{n-2}/u_n}. \tag{4.4}$$

Write $z_n = u_n/u_{n+1}$. Then (4) yields

$$\begin{aligned} z_{n-3} &= \frac{1}{n-2} + \frac{3a}{(n-2)(u_{n-2}/u_{n-1})(u_{n-1}/u_n)} \\ &= \frac{1}{n-2} + \frac{3a}{(n-2)\, z_{n-2} z_{n-1}}, \end{aligned} \tag{4.5}$$

or

$$z_n = \frac{1}{n+1} + \frac{3a}{(n+1)\, z_{n+1} z_{n+2}}, \qquad n = 0, 1, 2, \ldots . \tag{4.6}$$

We can then proceed as before to obtain upper and lower bounds. We have

$$\begin{aligned} z_n &> \frac{1}{n+1}, \qquad n = 0, 1, 2, \ldots \\ z_n &< \frac{1}{n+1} + \frac{3a(n+2)(n+3)}{(n+1)} \\ z_n &> \frac{1}{n+1} + \frac{3a/(n+1)}{\left(\frac{1}{n+2} + \cdots\right)\left(\frac{1}{n+3} + \cdots\right)}, \end{aligned} \tag{4.7}$$

and so forth.

## 5. Estimates for $u_0$

Using the foregoing technique, we can obtain estimates for $u_0/u_1 = z_0$ and $u_1/u_2 = z_1$. We also have

$$\begin{aligned} u_2 &= \int_0^\infty t^2 e^{-t} e^{-at^3}\, dt \\ &= (e^{-t})(-e^{-at^3}/3a)]_0^\infty - \frac{1}{3a} \int_0^\infty e^{-t-at^3}\, dt \\ &= \frac{1}{3a} - \frac{u_0}{3a}, \end{aligned} \tag{5.1}$$

or

$$u_0 = 1 - 3au_2 . \tag{5.2}$$

Hence, if

$$u_0/u_1 = z_1, \qquad \text{and} \qquad u_1/u_2 = z_2, \tag{5.3}$$

we have

$$u_2 = \frac{u_1}{z_2} = \frac{u_0}{z_1 z_2}. \tag{5.4}$$

Hence

$$\begin{aligned} u_0 &= 1 - \frac{3au_0}{z_1 z_2}, \\ u_0 &= \frac{1}{\left(1 + \frac{3a}{z_1 z_2}\right)}, \end{aligned} \tag{5.5}$$

whence upper and lower bounds for $z_1$ and $z_2$ yield upper and lower bounds for $u_0$.